Bifurcation Theory and Methods of Dynamical Systems

Luo Dingjun, Wang Xian, Zhu Deming & Han Maoan

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Bifurcation Theory and Methods of Dynamical Systems

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Preface

The bifurcation theory has been developed rapidly and is playing an ever more important role in nonlinear sciences and many applied fields during the past three decades. A lot of papers and a number of books have been published in this area. But up to now, to our knowledge, it seems that there are only a few books written by Chinese mathematicians to expound the bifurcation theory systematically. This is one of the main reasons why we have done this work. This book focuses on the bifurcation theory and methods of dynamical systems defined by ordinary differential equations, and tries to make the recent research results accessible to researchers who are interested in this subject. This is far from being a complete treatise on bifurcation theory, but a monograph which, together with the references quoted, does contain considerable information on the present state of the bifurcation theory of dynamical systems and introduces many relevant results obtained recently by the authors and some other Chinese mathematicians, which may not be known worldwide.

In Chapter 1, we provide a review of some basic concepts and general results in the theory of dynamical systems, which are the necessary background material for this book. We do not give any proof of the theorems and refer the reader to the related literature cited in our references.

Chapter 2 presents several kinds of limit cycle bifurcations of 2-dimensional systems, such as Hopf bifurcation in a very general version, multiple limit cycle bifurcation, Poincaré bifurcation and some of the general results on homoclinic and heteroclinic bifurcations obtained by the authors recently. The famous Hilbert’s 16th
problem concerned with the limit cycle problem of polynomial systems is studied from the bifurcation point of view. A number of examples are given to illustrate some interesting bifurcation diagrams of certain planar and toral differential systems.

Chapter 3 is devoted to the limit cycle problems in planar polynomial Liénard systems as well as some applied models. A number of theorems are given, concerning which the corresponding Liénard system has at most one or two limit cycles both with only one and with more than one critical points inside. By using these theorems, we obtain certain complete bifurcation diagrams for several kinds of polynomial Liénard systems. The results are completely global in the sense of both the phase plane and the parameter space.

From Chapter 4, we begin to consider the higher dimensional systems. Chapter 4 discusses the bifurcations of local periodic solutions for general periodic perturbed systems. The methods of Liapunov-Schmidt, averaging and integral manifold are developed for general $n$-dimensional non-autonomous systems. Then the planar systems with perturbation terms that are periodic in $t$ are studied. The bifurcations of invariant tori and subharmonic solutions near a center, a fine focus, or a periodic orbit of a planar system are studied in more detail.

Chapter 5 is concerned with Hopf bifurcation, degenerate Hopf bifurcation of periodic orbits, and bifurcation of homoclinic loop to a saddle with certain number of negative (or positive) eigenvalues for higher dimensional systems. The bifurcations of periodic orbits in resonance and degenerate cases, and of local invariant tori near a center point and global invariant tori near a large periodic orbit for certain systems are considered. The chaotic dynamics which appears in either the break of homoclinic loops or the weak Sil'nikov phenomenon is also considered. It serves to facilitate a comparatively intuitive understanding of the chaotic behavior.

The last chapter, Chapter 6, considers the persistence and transversality of homoclinic and heteroclinic loops. Theories of exponential trichotomy and Fenichel's invariant manifolds are introduced, and
then the principal normal coordinate system along the singular orbit and the Melnikov vector to measure the separation between stable and unstable manifolds of certain hyperbolic, nonhyperbolic equilibria and normally hyperbolic invariant manifolds are constructed.

In conclusion, we would particularly like to express our appreciation to Professor K. Shiraiwa for his recommendation of this book. We would like also to thank Professor Ye Yanqian for his help and encouragement to us to study dynamical systems. Lastly, we wish to thank the Foundations of the National Natural Science and the State Education Commission of China for the supports given to our research programs.
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Chapter 1

Basic Concepts and Facts

We are going to study the local and global behaviors of dynamical systems defined by differential equations

\[ \dot{x} = X(t, x), \]  

(1.0.1)

where \( t \) is the time, vector \( x \) is varied in an open set of \( \mathbb{R}^n \) or, more generally, a differentiable manifold. Usually, (1.0.1) is called a non-autonomous system. If \( X \) is independent of \( t \), then it is called an autonomous system.

In this chapter, we establish the concept of bifurcation and some related ideas needed for understanding the later chapters.

In Sec. 1.1 we first introduce the linearization of a given system near a critical point. Then we give the definition of hyperbolicity of a critical point (or a fixed point of a map) and of a periodic orbit (or a periodic point of a map). The notion of stable and unstable manifolds and transversal intersection are also introduced.

Section 1.2 is devoted to the concepts of globally topological equivalence, structural stability and the concept of bifurcation.

In Sec. 1.3 we discuss all the possible changes starting from a bifurcation system, which are concerned with the concepts of codimension and unfoldings.

In the last section we introduce the theory of center manifold.

For all the theorems related to some basic results of dynamical system theory introduced in this chapter we only give statements
Chapter 1. Basic Concepts and Facts

without proofs. If the readers are interested in some detailed proof, please see the related references.

1.1. Hyperbolicity and Transversality. $\lambda$-Lemma

1.1.1. Linearization. Hyperbolicity of critical point. Local invariant manifolds

Consider the autonomous system

$$\dot{x} = X(x), \quad x \in \mathbb{R}^n, \quad (1.1.1)$$

where $X: U \rightarrow \mathbb{R}^n$ is a $C^r$ map with $U$ an open set in $\mathbb{R}^n$ and $r \geq 1$. (1.1.1) is often called a $C^r$ vector field on $U$. Let $p$ be a critical point of (1.1.1). There is no harm in assuming $p$ to be the origin of $\mathbb{R}^n$, i.e., $X(0) = 0$ is satisfied. For the case that $p$ is a critical point of a vector field defined on a manifold, we may take a local coordinate system around $p$; then the vector field can also be represented by (1.1.1).

It is natural to consider the associated linear system of (1.1.1)

$$\dot{x} = Ax, \quad A = DX(0), \quad (1.1.2)$$

where $A$ is a constant $n \times n$ matrix. We call (1.1.2) a linearization of (1.1.1) at $O$. The geometrical structure of the solutions of linear system (1.1.2) is well known. Assume that $A$ has $s$ eigenvalues with negative real parts, $u$ eigenvalues with positive real parts, and $c$ eigenvalues with zero real parts, where $s + u + c = n$. $\mathbb{R}^n$ can be represented as the direct sum of the subspaces as follows.

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c,$$

where $E^s$, $E^u$, and $E^c$ are the eigenspaces corresponding to the eigenvalues with negative, positive and zero real parts, and are referred to as stable, unstable, and center subspaces respectively. The solutions of (1.1.2) with initial conditions contained in either $E^s$, $E^u$ or
$E^c$ must remain in that particular subspaces for all time $t$. Hence they are all examples of invariant manifolds, see Definition 1.1.1 below. Moreover, any solution starting in $E^s$ ($E^u$) approaches $x = 0$ as $t \to +\infty (-\infty)$.

**Definition 1.1.1.** Let $S \subset \mathbb{R}^n$ be a set, then $S$ is said to be invariant under system (1.1.1) if for any $x_o \in S$, the solution starting from $x_o$, $\varphi(t, x_o) \in S$ for all $t \in \mathbb{R}$. An invariant set with manifold structure is called an invariant manifold.

By introducing a suitable linear transformation, one may transform (1.1.2) into the block diagonal form

$$
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix} =
\begin{pmatrix}
A_s & 0 & 0 \\
0 & A_u & 0 \\
0 & 0 & A_c
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix},
$$

(1.1.3)

where $(u, v, w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$, and $A_s$, $A_u$, $A_c$ are $s \times s$, $u \times u$ and $c \times c$ matrices with eigenvalues of negative, positive and zero real parts, respectively. Correspondingly, (1.1.1) can be represented in the form

$$
\begin{align*}
\dot{u} &= A_s u + U(u, v, w), \\
\dot{v} &= A_u v + V(u, v, w), \\
\dot{w} &= A_c w + W(u, v, w),
\end{align*}
$$

(1.1.4)

where $U$, $V$, $W$ and all their first order partial derivatives vanish at $(0, 0, 0)$. The critical point $x = 0$ of (1.1.4) possesses a $C^r$ s-dimensional local stable manifold $W^{s}_{loc}(0)$, $u$-dimensional local unstable manifold $W^{u}_{loc}(0)$ and $c$-dimensional local center manifold $W^{c}_{loc}(0)$.

These manifolds are all local invariant manifolds. We shall state some results about them in the following paragraphs.

**Definition 1.1.2.** If $c = 0$, i.e., none of the eigenvalues of $A$ have zero real parts, then $x = 0$ is called a hyperbolic critical point (both of (1.1.1) and (1.1.2)).
Chapter 1. Basic Concepts and Facts

It is reasonable to ask: whether the behavior near the critical point \( x = 0 \) of (1.1.1) can be represented by the behavior of the linearization (1.1.2), which is concerned with the concept of locally topological equivalence.

**Definition 1.1.3.** We say that two vector fields \( X, Y : U \to \mathbb{R}^n \) with a critical point \( O \in U \) (open set) \( \subset \mathbb{R}^n \) are locally topologically equivalent at \( O \) if there exists a homeomorphism \( h \) of a neighborhood \( V \subset U \) of \( O \) onto a neighborhood \( W \) of \( O \), which takes the orbits of \( X \) in \( V \) to the orbits of \( Y \) in \( W \) preserving their orientation. This means that there exist \( t, t' > 0 \) such that for the flows \( \varphi, \psi \) of \( X, Y \), the relation

\[
h \varphi_t(p) = \psi_{t'} h(p)
\]

holds for all \( p \in U \).

An important theorem about the locally topological equivalence of (1.1.1) and (1.1.2), proved independently by Hartman and Grobman, is the following

**Theorem 1.1.1.** If \( O \) is a hyperbolic critical point of (1.1.1), then the vector fields (1.1.1) and (1.1.2) are locally topologically equivalent at \( O \).

**Proof.** See [79] or [115].

About the existence of local manifolds \( W^s_{\text{loc}} \) and \( W^u_{\text{loc}} \), we have the following theorem. For the proof, see [115], [128].

**Theorem 1.1.2.** Let \( X : U \to \mathbb{R}^n \) be a \( C^r \) vector field with \( O \) a hyperbolic critical point. Then there exist a \( C^r \) s-dimensional local stable manifold \( W^s_{\text{loc}}(O) \) and a \( C^r \) u-dimensional local unstable manifold \( W^u_{\text{loc}}(O) \), which are tangent to the invariant subspaces \( E^s \) and \( E^u \) at the origin respectively.

The orbit structure near a hyperbolic critical point is shown in Fig. 1.1.1. We remark that (a) corresponds to the particular case \( s = 0 \), for which \( O \) is called a source, and (c) corresponds to the particular
case \( u = 0 \), for which \( O \) is called a sink. The local stable and unstable manifolds can be represented by the following sets respectively:

\[
W_{\text{loc}}^s(O) = \{ x \in U \mid \varphi_t(x) \to O \text{ as } t \to \infty \},
\]

\[
W_{\text{loc}}^u(O) = \{ x \in U \mid \varphi_t(x) \to O \text{ as } t \to -\infty \},
\]

We now give some remarks for the case of discrete systems. An important idea is to reduce the study of continuous time system to the study of an associated discrete time system (diffeomorphism), which is originally due to H. Poincaré. A CT discrete system means a CT diffeomorphism

\[
f : U \to \mathbb{R}^n, \quad U \text{ an open set in } \mathbb{R}^n.
\]

The orbit passing through \( p \) of \( f \) is the set \( \{ f^i(p) \}_{i=0, \pm 1, \pm 2, \ldots} \). If \( f(0) = 0 \), then \( O \) is called a fixed point of \( f \). Associated to (1.1.6), we consider the linear map

\[
x \mapsto Ax, \quad A = Df(O),
\]

which also has the invariant subspaces \( E^s \), \( E^u \) and \( E^c \), where \( s + u + c = n \) and \( A \) has \( s \), \( u \) and \( c \) eigenvalues with moduli less than, greater than and equal to one, respectively. Parallel to the case of vector fields, we have the following definitions.

**Definition 1.1.4.** If \( c = 0 \), i.e., none of the eigenvalues of \( A \) has modulus one, then \( x = 0 \) is called a hyperbolic fixed point (both of (1.1.6) and (1.1.7)).
Definition 1.1.5. Two maps \( f, g : U \rightarrow \mathbb{R}^n \) with fixed point \( O \in U \) are locally topologically conjugate at \( O \) if there exists a neighborhood \( V \subset U \) of \( O \) and a homeomorphism \( h \) on \( V \) such that

\[
h \circ f = g \circ h, \quad \text{on } V.
\]

Essentially, everything about local stable and unstable manifolds for critical points of vector field holds for fixed points of map (1.1.6).

1.1.2. Hyperbolicity of periodic solutions. Global stable and unstable manifolds

For diffeomorphism \( f \), if there is a point \( p \) and integer \( k \geq 1 \) such that \( f^k(p) = p \), then \( p \) is said to be a \( k \)-periodic point of \( f \). The least of such \( k \) is called the period of \( p \). It is seen that \( p \) can be regarded as a fixed point of map \( f^k \).

Definition 1.1.6. We say that \( p \) is a hyperbolic \( k \)-periodic point of \( f \), if \( p \) is a hyperbolic fixed point of \( f^k \).

At this time, \( \{p, f(p), \cdots, f^{k-1}(p)\} \) is called a hyperbolic periodic orbit. It is easy to see that every point \( f^i(p) \) is hyperbolic for \( i = 0, 1, \cdots, k-1 \), for which we have local stable and unstable manifolds as shown in Sec. 1.1.1. Transporting them around the orbit by applying

Fig. 1.1.2
1.1. Hyperbolicity and Transversality. \( \lambda \)-Lemma

For a periodic orbit \( \gamma \) of a vector field \( X \), we take a point \( p \in \gamma \) and a \((n-1)\)-dimensional section hyperplane \( \Sigma \) passing through \( p \) and transversal to \( \gamma \). Let \( V \) be a sufficiently small neighborhood of \( p \) on \( \Sigma \), then the forward orbit through each point \( x \in V \) must meet \( \Sigma \) again at a time \( \tau \) close to \( T \) (the period of \( \gamma \)). Then we can define a Poincaré map

\[
P: V \to \Sigma \quad \text{by} \quad P(x) = \varphi_\tau(x),
\]

with \( P(p) = p \). By using the implicit function theorem it can be shown that \( P \) is of class \( C^r \) if \( X \) is. Similarly we may construct the inverse \( P^{-1} \) by following the orbit backward instead of forward. Thus \( P \) is a local \( C^r \) diffeomorphism.
**Definition 1.1.7.** If \( p \) is a hyperbolic fixed point of \( P \), \( \gamma \) is called a hyperbolic periodic orbit of \( X \).

It is easy to see that this definition is independent of the choice either of \( \Sigma \) transversal to \( \gamma \) or of \( p \) on \( \gamma \). As a hyperbolic fixed point of \( P \), \( p \) has local stable and unstable manifolds as above. Let \( p \) move around \( \gamma \) and put these manifolds together to constitute the corresponding local stable and unstable manifolds of \( \gamma \), see Fig. 1.1.3. We may present them by

\[
W_{\text{loc}}^s(\gamma) = \bigcup_{p \in \gamma} W_{\text{loc}}^s(p)
\]

and

\[
W_{\text{loc}}^u(\gamma) = \bigcup_{p \in \gamma} W_{\text{loc}}^u(p).
\]

Let \( p \) be a hyperbolic critical point of vector field \( X \) on \( U \subset \mathbb{R}^n \). The set

\[
W^s(p) = \bigcup_{t \geq 0} \varphi_{-t} W_{\text{loc}}^s(p)
\]

is called the **global stable manifold** of \( p \), i.e.,

\[
W^s(p) = \{ x \in U | \omega(x) = p \},
\]

where \( \omega(x) \) and \( \alpha(x) \) below are the \( \omega \), \( \alpha \) limit sets of \( x \) respectively. Because \( \varphi_t(x) \to p \) as \( t \to \infty \), \( \varphi_t(x) \in W_{\text{loc}}^s(p) \) as \( t \) large enough, or equivalently \( x \in \varphi_{-t} W_{\text{loc}}^s(p) \), then we may get \( W^s(p) \) by taking the union of \( \varphi_{-t} W_{\text{loc}}^s(p) \) over all \( t \geq 0 \).

It must be stressed that \( W^s(p) \) may not be a submanifold of \( \mathbb{R}^n \). For example, the double loops passing through a saddle point \( p \) for a planar vector field is not a manifold, see Fig. 1.1.4.

Similarly we define the **global unstable manifold** of a fixed point \( p \) by

\[
W^u(p) = \{ x \in U | \alpha(x) = p \} = \bigcup_{t \geq 0} \varphi_t W_{\text{loc}}^u(p).
\]

An analogous discussion applies to \( C^r \) diffeomorphism \( f \): define the global stable and unstable manifolds \( W^s(p) \), \( W^u(p) \) of a hyperbolic